A quick survey on even-hole-free graphs

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Even-hole-free graphs (or ehf graphs)

- H is an induced subgraph of G if H can be obtained from G by *deleting vertices*
- G is H-free if no induced subgraph of G is isomorphic to H
- When $\mathcal{F}$ is a family of graphs, $\mathcal{F}$-free means H-free, $\forall H \in \mathcal{F}$
- Even hole: induced cycle of even length (i.e. no chord in the cycle)
- G is even-hole-free means G does not *contain* an even hole
- Some examples: chordal graphs, complete graphs

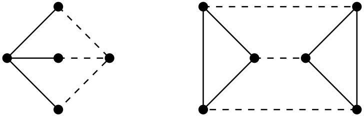


Figure: Theta and prism

Remark. (Theta, prism)-free is a superclass of even-hole-free

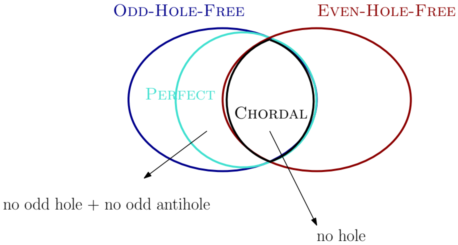
Motivation

Perfect graphs

- G is perfect if $\chi(H) = \omega(H)$, for any H induced subgraph of G
- Strong Perfect Graph Theorem: G is perfect iff G contains no *odd hole* & no *odd antihole*

Even-hole-free graphs

- even-hole-free = no even hole & no antihole of length ≥ 6

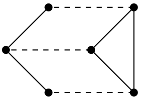


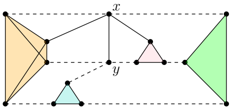
Decomposition of even-hole-free graphs

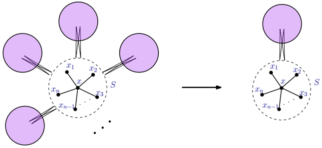
Theorem (da Silva & Vušković)

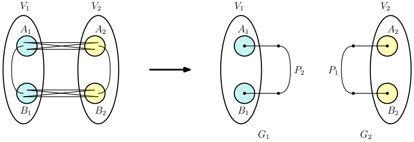
A connected even-hole-free graph is either basic, or it has a 2-join or a star cutset.

Basic graphs: cliques, holes, long pyramids, nontrivial basic graphs









Motivation

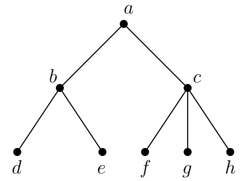
- A sort of dichotomy between "even-hole-free graphs" and "perfect graphs"

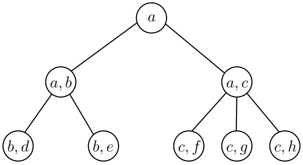
	ehf graphs	Perfect graphs
Structure	"Simpler"	More complex
Polynomial α, χ	?	YES

- Better understanding of the structure of even-hole-free graphs

Treewidth

Tree decomposition





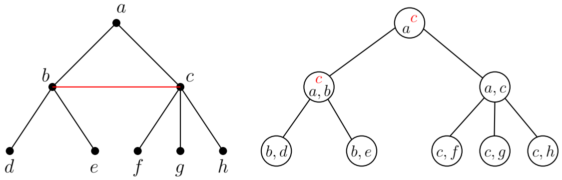
AXIOMS

1. Every vertex is in a bag
2. Every edge is in a bag
3. $\forall v \in V(G)$, the support of v forms a subtree

- Treewidth of G (or $tw(G)$) measures how close G from being a tree
- Tree decomposition of G: "gluing" the pieces of subgraphs of G in a tree-like fashion (a tree decomposition resembles "fat tree" with nodes represented as "bags" of vertices)
  - width of T = the size of the largest bag - 1
  - treewidth of G: width of the optimal tree decomposition of G

Treewidth

Tree decomposition



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Algorithmic use of treewidth

Many graph optimization problems that are NP-hard become tractable on bounded treewidth graphs

Theorem (Courcelle, 1990)

Every graph property definable in the monadic second-order logic (MSO) formulas can be decided in linear time on class of graphs of bounded treewidth.

Some graph problems expressible in MSO:

- maximum independent set, maximum clique, coloring

Treewidth of even-hole-free graphs

Observation: since complete graph is ehf, the treewidth of the class is unbounded

- When *planar* $\rightarrow tw \leq 49$ [Silva, da Siva, Sales, 2010]
- Pan-free $\rightarrow tw \leq 1.5\omega(G) - 1$ [Cameron, Chaplick, Hoàng, 2015]
- K_3-free $\rightarrow tw \leq 5$ [Cameron, da Silva, Huang, Vušković, 2018]
- Cap-free $\rightarrow tw \leq 6\omega(G) - 1$ [Cameron, da Silva, Huang, Vušković, 2018]

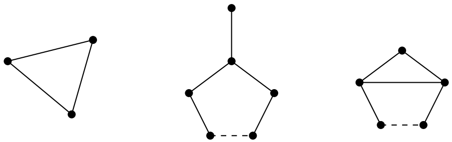

Figure: Triangle, pan, and cap

Ehf graphs of unbounded "width"

Diamond-free ehf has unbounded *rank-width* (implies unbounded treewidth) [Adler, Le, Müller, Radovanović, Trotignon, Vušković, 2017]

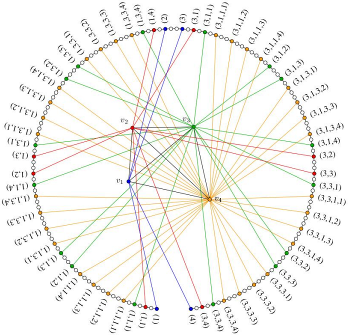

Figure: A diamond-free ehf graph of large *rank-width*; it contains large clique

Question: **What if the clique size is bounded?**

Ehf graphs with no K_4

Recall: ehf graphs with $\omega = 2$ have treewidth at most 5. What about $\omega = 3$?

Cameron, Chaplick, and Hoáng (2018) ask the following:
Is the treewidth of ehf graph (in general) bounded by a function of its max clique size?

Ehf graphs with no K_4

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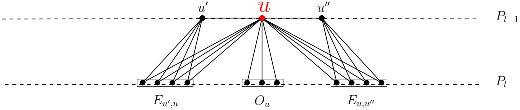
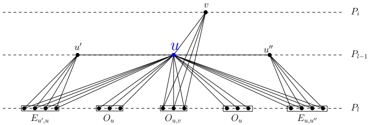
Cameron, Chaplick, and Hoáng (2018) ask the following:
Is the treewidth of ehf graph (in general) bounded by a function of its max clique size?

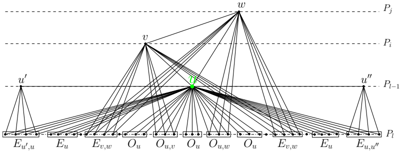
- S., Trotignon (2019): Ehf graphs with no K_4 can have arbitrarily large treewidth.
- So, bounded clique size does not imply bounded treewidth.

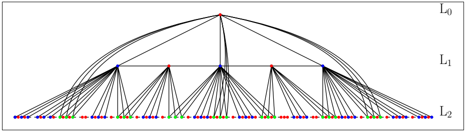
Ehf graphs of unbounded treewidth

- A family of K_4-free graphs with arbitrarily large tw

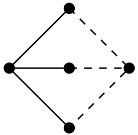




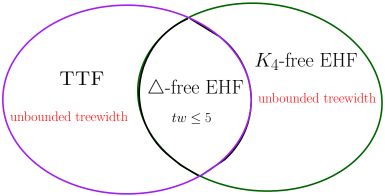




Theta-free graphs and even-hole-free graphs

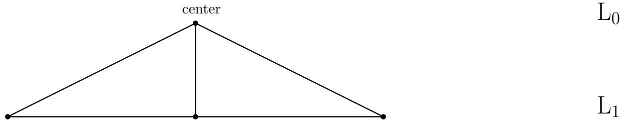


- A theta is a graph induced by three paths s.t. any two of them induce a hole.
- Theta-free graphs is a superclass of even-hole-free graphs.



$G(\ell, k)$, with $\ell = 2$ and $k = 4$

Layered wheel ((theta, triangle)-free) construction



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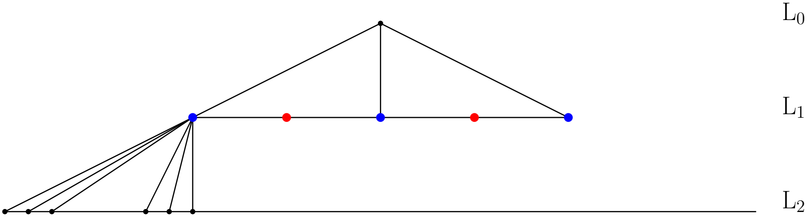

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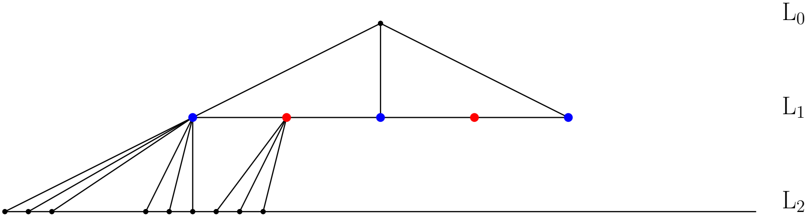

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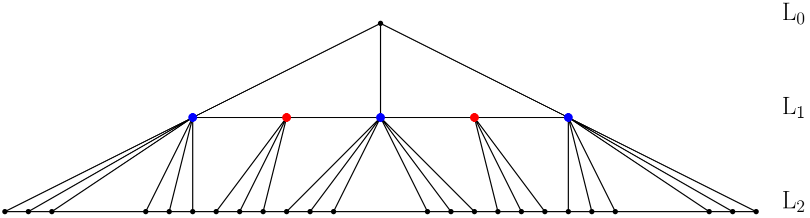

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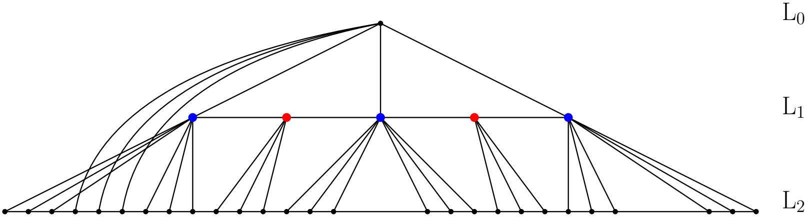

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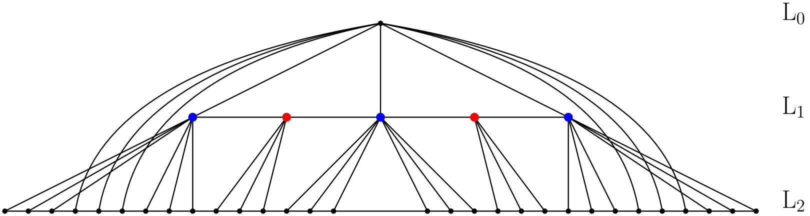

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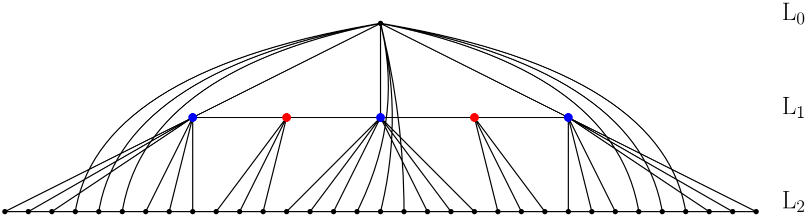

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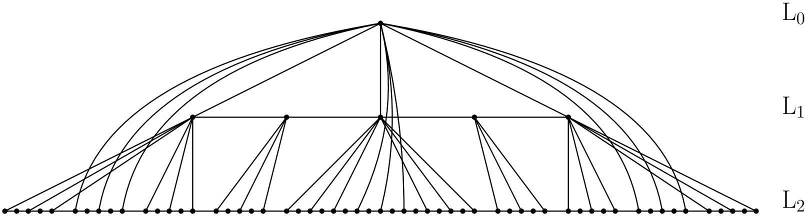

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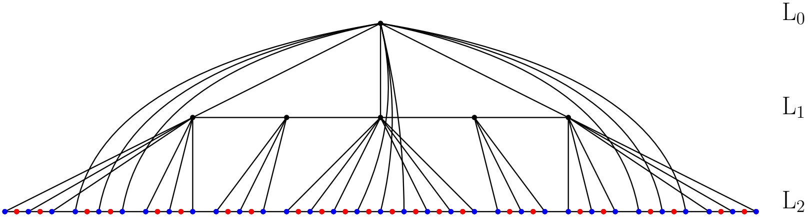
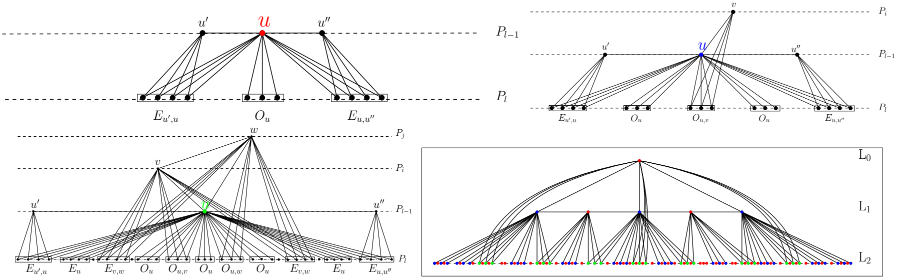


$G(\ell, k)$, with $\ell = 2$ and $k = 4$

Ehf graphs of unbounded treewidth

Bounded clique size does not imply bounded treewidth

- A family of K_4-free graphs with arbitrarily large tw



A bound on the treewidth of layered wheels

Theorem (S., Trotignon (2019))

The treewidth of layered wheel is in $\mathcal{O}(\log(n))$ *where* n *is the vertex size.*

A bound on the treewidth of layered wheels

Theorem (S., Trotignon (2019))

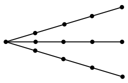
The treewidth of layered wheel is in $\mathcal{O}(\log(n))$ where n is the vertex size.

Conjecture

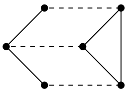
- *There exists a constant c such that for any (theta, triangle)-free graph G, we have*
$$tw(G) \leq c \log |V(G)|.$$
- *Idem for K_4-free ehf graph.*

Excluding more structure

- For $k \geq 1$, a (theta, triangle, $S_{k,k,k}$)-free graph G has treewidth at most $\mathcal{O}(k^6)$.
- For $k \geq 1$, an (even hole, pyramid, K_t, $S_{k,k,k}$)-free graph G has treewidth at most $\mathcal{O}(t^{10}k^9)$.



$S_{k,k,k}$ $k = 5$



pyramid

Another observation from layered wheels

Even-hole-free graphs with no K_4 have unbounded treewidth

- Our construction which certifies this contains large clique minor
- It also contains vertices of high degree

Are these two conditions necessary?

Another observation from layered wheels

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- It also contains vertices of high degree

Are these two conditions necessary? YES!

- Even-hole-free graphs with no clique minor have bounded treewidth [Aboulker, Adler, Kim, S., Trotignon, 2020]
- Even-hole-free graphs of bounded degree have bounded treewidth [Abrishami, Chudnovsky, Vušković, 2020]

Even-hole-free graphs with no H-minor

Theorem (Aboulker, Adler, Kim, S., Trotignon, 2020)

Even-hole-free graphs with no H-minor for some graph H have bounded treewidth. (This is actually proven for (theta, prism)-free graphs.)

- This provides another proof that planar ehf graphs have bounded treewidth.
- For the proof, we develop an "induced wall theorem" for graphs excluding fixed minor.
- From this, we derive that ehf graphs excluding fixed minor have bounded treewidth.

Even-hole-free graphs with no H-minor

Theorem (Induced wall theorem for graphs excluding H-minor)

If G is H-minor-free with $tw(G) \geq f_H(k)$, then G contains a $(k \times k)$-wall or the line graph of a chordless $k \times k$-wall as an induced subgraph.

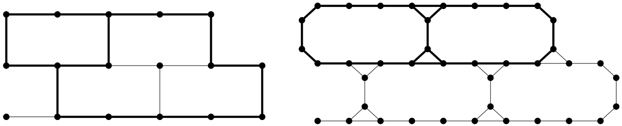


Figure: A (3×3)-wall and the line graph of chordless (3×3)-wall

Even-hole-free graphs with no H-minor

Theorem (Fomin, Golovach, Thilikos, 2011)

For every H, there exists a constant $c_H > 0$ and an integer k s.t. for every connected H-minor free graph G with $tw(G) \geq c_H \cdot k^2$, G contains either Γ_k or Π_k as a contraction.

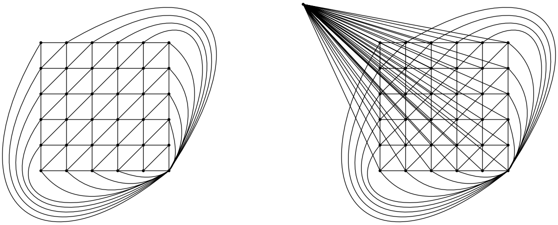


Figure: Γ_6 and Π_6

Even-hole-free graphs of bounded degree

Conjecture (Aboulker, Adler, Kim, S., Trotignon, 2020)

Even-hole-free graphs with bounded degree have bounded treewidth.

We prove the following cases:

- **Subcubic ehf graphs** have treewidth at most 3
  - Approach: a full structure theorem for subcubic (theta, prism)-free graphs (every graph is either simple or it has a "nice" separator which yields boundedness on the treewidth).
- **Pyramid-free ehf graphs of degree** ≤ 4
  - Approach: a combination of structural properties to show K_6-minor-freeness.
  - $tw(G) \leq f_{K_6}(3)$, with f as in the induced grid theorem.

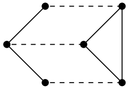


Figure: Pyramid

Structure theorem of subcubic even-hole-free graphs

Theorem (Aboulker, Adler, Kim, S., Trotignon, 2020)

Let G be a (theta, prism)-free subcubic graph. Then either:

- *G is a basic graph; or*
- *G has a clique separator of size at most 2; or*
- *G has a proper separator.*

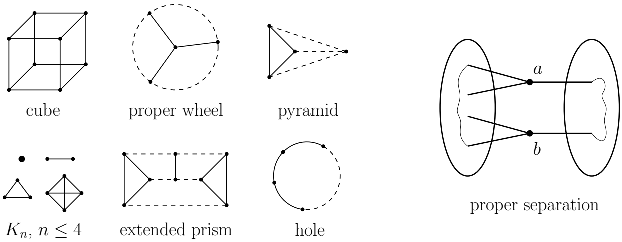


Figure: Basic graphs and proper separator

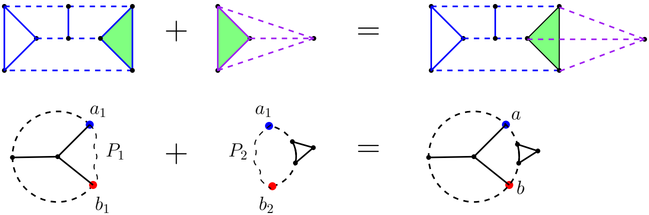
Treewidth of even-hole-free graphs (a proof)

Theorem (Aboulker, Adler, Kim, S., Trotignon, 2020)

Subcubic even-hole-free graphs have treewidth ≤ 3.

Sketch of proof.

- Every basic graph has treewidth at most 3.
- "Gluing" along a clique and proper gluing preserve treewidth to be ≤ 3.



Tw of (even hole, pyramid)-free graphs of max degree 4

Recall that we prove the following:

Theorem (Aboulker, Adler, E. Kim, S., Trotignon, 2020)

Every (even hole, pyramid)-free graph of maximum degree 4 has treewidth $< f_{K_6}(3)$.

- f is the bound given in the 'induced grid theorem'
- The core of the proof: If G is (even hole, pyramid)-free graph of maximum degree at most 4, then G contains no K_6-minor.

Theorem (S., Trotignon, 2021+)

(Even hole, pyramid)-free graphs of max degree 4 have treewidth ≤ 4.

Sketch of proof. Similar to the subcubic case, with more basic graphs.

Even-hole-free graphs of bounded degree

The "bounded degree $\Rightarrow$ bounded treewidth" conjecture has been proven! (using another technique: balanced separator)

Theorem (Abrishami, Chudnovsky, Vušković, 2020)

Ehf graphs of bounded degree have bounded treewidth. (This is actually proven for a superclass of ehf graphs.)

Open problems

Motivation: grid-minor theorem of Robertson and Seymour
There is a function f such that if $tw(G) > f(k)$, then G contains (as an induced subgraph) one of the following:

- a subdivision of a $(k \times k)$-wall
- line graph of a subdivision of a $(k \times k)$-wall
- a vertex of degree at least k

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